Heavy-Traffic Analysis of the M/G/1 Queue With Priority Classes

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Heavy-Traffic Analysis of the $M/G/1$ Queue with Priority Classes

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Abstract

We consider the $M/G/1$ queue with two priority classes, for the case that at least one of the service time distributions is regularly varying of index $-\nu$ with $1 < \nu < 2$. It is shown for this heavy-tailed case that the waiting time distribution of the low-priority customers is regularly varying of index one degree higher than that of the service time distribution with the heaviest tail. We also prove a heavy-traffic limit theorem for the steady-state low-priority waiting time W_2. When the low-priority traffic load $\rho_2 \to 1 - \rho_1$ (ρ_1 being the high-priority traffic load), the contracted low-priority waiting time $\Delta(\rho_2)W_2/\beta_1$ converges in distribution to the Mittag-Leffler distribution $R_{\nu-1}(t)$ where $\Delta(\rho_2)$ is a particular function of ρ_2 with the property that $\Delta(\rho_2) \to 0$ for $\rho_2 \to 1 - \rho_1$, $R_{\nu-1}(t)$ is a proper distribution with Laplace-Stieltjes transform $1/(1 + s^{\nu-1})$ and β_1 is the mean of class-1 service time. The heavy-traffic limit theorem gives rise to an approximation for the steady-state distribution of W_2, which is extensively tested numerically.

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1 Introduction

We consider the $M/G/1$ queueing model with two priority classes, with either the nonpreemptive or the preemptive resume discipline. We are interested in the effect of the priority structure on the tail of the low-priority waiting-time distribution.

Let us first introduce some notations. The high-priority class is indexed by 1 and the low-priority class by 2. Let $B_j(t)$ denote the service time distribution function of class-j, λ_j the arrival rate of class-j and ρ_j the traffic load of class-j for $j = 1, 2$. The arrival processes of the two classes are independent. For $j = 1, 2$, put

$$\begin{aligned}
\beta_j &:= \int_0^\infty t \mathrm{d}B_j(t) < \infty, \\
\beta_{j2} &:= \int_0^\infty t^2 \mathrm{d}B_j(t) \le \infty, \\
\rho_j &:= \lambda_j \beta_j, \\
\rho &:= \rho_1 + \rho_2,
\end{aligned}$$

and assume that $\rho < 1$.

Let W_2 denote the steady-state waiting time of the low-priority customers until start of the service (note that it has the same distribution for the nonpreemptive and the preemptive resume discipline). When $\beta_{j2} < \infty$ for $j = 1, 2$, the following heavy-traffic limit theorem for W_2 holds (cf. [1]):

$$\lim_{\rho_2 \uparrow 1-\rho_1} \Pr\{\Delta W_2 \leq t\} = 1 - e^{-t}, \ t \geq 0, \tag{1}$$

where $\Delta := \frac{2(1-\rho_1)(1-\rho)}{\rho_1\beta_{12}/\beta_1 + \rho_2\beta_{22}/\beta_2}$.

In the present study we prove a heavy-traffic limit theorem for the steady-state low-priority waiting time distribution in the $M/G/1$ queue with two priority classes when β_{12} and/or β_{22} is *not* finite. More specifically, we assume that at least one of the service time distributions has a regularly varying tail with index $-\nu$, i.e.

$$1 - B_j(t) \sim L(t)t^{-\nu} \text{ as } t \to \infty, \tag{2}$$

for $j = 1$ and/or $j = 2$, where $L(t)$ is a slowly varying function and $1 < \nu < 2$. Here $f(t) \sim g(t)$ as $t \to \infty$ stands for $\lim_{t\to\infty} f(t)/g(t) = 1$. A measurable positive function $L(t)$ defined on some neighborhood $[a, \infty)$ is called a slowly varying function if for all $x > 0$, $\lim_{t\to\infty} L(xt)/L(t) = 1$.

Our motivation for this study, apart from the wish to extend the heavy-traffic limit theorem, is the following. Plots of recent traffic measurements in Ethernet Local Area Networks [21], Wide Area Networks [19] and VBR video [2] have shown a striking similarity when one considers a time period of hours, minutes or milliseconds: bursty subperiods are alternated by less bursty subperiods on each time scale. This scale-invariant or self-similar feature of traffic, and the related phenomenon of *long-range dependence* (i.e., the integral of the covariance of the traffic rate diverges), were convincingly demonstrated in [17] via a careful statistical analysis. These phenomena may have a profound effect on system performances and therefore deserve a detailed analysis. In several recent studies, cf. [7, 5], it has been pointed out that fluid or ordinary queues with heavy-tailed input distributions (like activity period distributions in a fluid queue, or service time distributions in an ordinary queue) are useful and tractable models for analyzing the effect of such traffic on system performance. An important and useful class of heavy-tailed distributions is the class of regularly varying distributions with index $-\nu$ where $1 < \nu < 2$, as specified in (2).

In [9] it has already been shown for the $GI/G/1$ queue with FCFS service discipline that the waiting time distribution has a regularly varying tail of index $1 - \nu$ if and only if the service time distribution has a regularly varying tail of index $-\nu$. Only recently the effect of priority disciplines on the waiting time tail behavior has been considered in the case of regularly varying service time distributions. It is shown in [22] that in the processor sharing $M/G/1$ queue the sojourn time distribution has a regularly varying tail with the same index as the service time distribution tail. In [5] a similar result is obtained for LCFS Preemptive Resume, whereas LCFS nonpreemptive leads to regularly varying sojourn time tails of index one higher than the service time tail (like FCFS). In communication networks often different traffic types can be distinguished, with different traffic characteristics and different performance requirements. It is then natural to impose a priority structure. Abate and Whitt [1] consider an $M/G/1$ queue with two priority classes and either the nonpreemptive or the preemptive resume discipline. They study, a.o., the effect of the service time distribution tails on the tails of the waiting time distributions. In this paper we consider the same model. We are mainly interested in the heavy-traffic situation.

The main result in this paper is a heavy-traffic limit theorem for the distribution of W_2 in the $M/G/1$ queue with two priority classes and at least one regularly varying service time distribution. It states that the contracted low-priority waiting time $\Delta(\rho_2)W_2/\beta_1$ converges in distribution to $R_{\nu-1}(t)$,

$$\lim_{\rho_2 \uparrow 1-\rho_1} \Pr\{\Delta W_2/\beta_1 \geq t\} = 1 \Leftrightarrow R_{\nu-1}(t), \ t \geq 0, \tag{3}$$

where $R_{\nu-1}(t)$ is a probability distribution with Laplace-Stieltjes (L-S) transform $1/(1+s^{\nu-1})$, and where Δ is the unique solution to the equation

$$\frac{K\left(\frac{\Delta}{1-\rho_1}\right)^{\nu-1} L\left(\frac{1}{\Delta}\right)}{1 \Leftrightarrow \rho} = 1, \tag{4}$$

with the property that $\Delta \downarrow 0$ for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$. Note that $\Delta = (1 \Leftrightarrow \rho_1)\left(\frac{1-\rho}{K}\right)^{\frac{1}{\nu-1}}$ when $L(t) \equiv 1$. Here K depends on whether the tail of $B_1(t)$ or of $B_2(t)$ is heavier, and $\Leftrightarrow \nu$ is the index of the heaviest of the two tails, i.e., the heaviest tail determines the tail of the (heavy-traffic) low-priority waiting time distribution.

This paper is organized as follows.

In Section 2 we characterize the service time distributions $B_j(\cdot)$ for $j = 1, 2$. We assume that one of the service time distributions has a regularly varying tail of index $\Leftrightarrow \nu$, another one has less heavy tail, or both of them have a regularly varying tail of the same index $\Leftrightarrow \nu$ where $1 < \nu < 2$.

In Section 3 we derive the asymptotic expansions of the L-S transforms of the service time distributions and the class-1 busy period distribution according to the assumptions in Section 2.

A representation for $w_2\{s\}$, the L-S transform of the distribution of the class-2 waiting time W_2, given by Abate and Whitt [1] is used in Section 4 to derive the asymptotic expansion of $w_2\{s\}$ for $s \downarrow 0$. It is also shown that the class-2 waiting time distribution $W_2(t)$ has a regularly varying tail of index $1 \Leftrightarrow \nu$ if the service time distributions satisfy the assumptions given in Section 2.

The aim of Section 5 is to show a reversed result; i.e., if $W_2(t)$ has a regularly varying tail with index $1 \Leftrightarrow \nu$ where $\nu > 1$ and it is known that the class-1 service time distribution $B_1(t)$ has a tail which is less heavy than $t^{-\nu}$, then the class-2 service time distribution $B_2(t)$ is regularly varying with index $\Leftrightarrow \nu$.

The asymptotic expansions obtained in Section 4 are used in Section 6 to derive the main result in this paper (the heavy-traffic limit theorem) which is presented above.

In Section 7 we generalize the heavy-traffic limit theorem for the waiting time distribution of the lowest priority class to the $M/G/1$ queueing model with k ($k \geq 2$) priority classes. We obtain a similar result as the above mentioned heavy-traffic limit theorem.

In Section 8 we make a comparison with a heavy-traffic limit theorem for the waiting time distribution in the $M/G/1$ queueing model *without* priority classes. This suggests approximating $\Pr\{W_2 > t\}$ by $\Pr\{(1 \Leftrightarrow \rho_1)W > t\}$, where W is the steady-state waiting time in the model without priority classes.

In Section 9 we propose an approximation for $\Pr\{W_2 > t\}$ based on the obtained heavy-traffic limit theorem, and we numerically investigate its accuracy as well as that of $\Pr\{(1 \Leftrightarrow \rho_1)W > t\}$. Both appear to perform very well over a wide range of ρ- and t-values.

2 On the service time distributions

In this section we describe the classes of distributions $B_1(\cdot)$ and $B_2(\cdot)$ for which we analyze the heavy traffic behavior of the low-priority waiting time distribution. For $s \geq 0$ and $j = 1, 2$, define the L-S transforms of the service time distributions and of the residual service time distributions,

$$\beta_j\{s\} \;:=\; \int_0^\infty e^{-st}\mathrm{d}B_j(t), \tag{5}$$

$$\beta_{je}\{s\} \;:=\; \frac{1}{\beta_j}\int_0^\infty e^{-st}\left(1 - B_j(t)\right)\mathrm{d}t. \tag{6}$$

Concerning the service time distributions $B_j(\cdot)$ for $j = 1, 2$, we only introduce assumptions about their tails, i.e. about $1 - B_j(t)$ for $t \to \infty$. It is assumed that one of the service time distributions has a regularly varying tail behavior, another one has less heavy tail behavior, or both of the service time distributions have a regularly varying tail with the same index. That is, one of the following assumptions holds,

$$(i) \quad 1 - B_1(t) \sim -\frac{1}{\Gamma(1-\nu)}(t/\beta_1)^{-\nu}L(t/\beta_1) \quad \text{as } t \to \infty, \tag{7}$$

$$M_{2\mu} := \int_0^\infty t^\mu \mathrm{d}B_2(t) < \infty \qquad \text{for a } \mu > \nu;$$

$$(ii) \quad 1 - B_2(t) \sim -\frac{1}{\Gamma(1-\nu)}(t/\beta_2)^{-\nu}L(t/\beta_2) \quad \text{as } t \to \infty,$$

$$M_{1\mu} := \int_0^\infty t^\mu \mathrm{d}B_1(t) < \infty \qquad \text{for a } \mu > \nu;$$

$$(iii) \quad 1 - B_j(t) \sim -\frac{1}{\Gamma(1-\nu)}(t/\beta_j)^{-\nu}L_j(t/\beta_j) \quad \text{as } t \to \infty,$$

$$L(t) := L_1(t) \text{ for } t \geq 0,$$

$$\alpha := \lim_{t\to\infty}\frac{L_2(t)}{L(t)} < \infty;$$

$$(iv) \quad 1 - B_j(t) \sim -\frac{1}{\Gamma(1-\nu)}(t/\beta_j)^{-\nu}L_j(t/\beta_j) \quad \text{as } t \to \infty,$$

$$L(t) := L_2(t) \text{ for } t \geq 0,$$

$$\alpha := \lim_{t\to\infty}\frac{L(t)}{L_1(t)} = \infty,$$

where $1 < \nu < 2$, $L(\cdot)$, $L_1(\cdot)$ and $L_2(\cdot)$ are slowly varying functions. To obtain our heavy-traffic limit theorem, we assume that $L(t)$ is continuous for sufficiently large t. Without loss of generality, we may assume $\nu < \mu < 2$.

3 Preliminaries

In this section we study some properties of the L-S transforms of the service and busy period distribution. These properties are implied by Assumption (7). As usual,

$$f(s) = \sum_{k=0}^{n} g_k(s) + \mathrm{o}(g_n(s)) \text{ as } s \downarrow 0$$

means that

$$\lim_{s\downarrow 0} \frac{f(s) \Leftrightarrow \sum_{k=0}^{n} g_k(s)}{g_n(s)} = 0,$$

and

$$f(s) = \sum_{k=0}^{n} g_k(s) + \mathrm{O}(g_n(s)) \text{ as } s \downarrow 0$$

means that

$$\limsup_{s\downarrow 0} \frac{f(s) \Leftrightarrow \sum_{k=0}^{n} g_k(s)}{g_n(s)} < \infty.$$

It is of particular interest that the tail behavior of the service time distributions is regularly varying with index $\Leftrightarrow\nu$, $\nu > 1$. The next lemma (cf. Lemma 2.2 of [8] and the lines following it) characterizes the tail behavior of a probability distribution in terms of its L-S transform.

Lemma 1 *Let X be a non-negative random variable with L-S transform $f(s)$.*
(i) If X has finite moments ϕ_k of order k, $k = 0, 1, ..., n$, then

$$f_n(s) := (\Leftrightarrow 1)^{n+1} \left(f(s) \Leftrightarrow \sum_{j=0}^{n} \phi_j \frac{(\Leftrightarrow s)^j}{j!} \right) = \mathrm{o}(s^n), \;\; s \downarrow 0. \tag{8}$$

(ii) If there exist constants a_j, $j = 0, ..., n$, such that

$$f(s) \Leftrightarrow \sum_{j=0}^{n} a_j s^j = \mathrm{o}(s^n), \;\; s \downarrow 0,$$

then $\phi_j < \infty$ and $\phi_j = (\Leftrightarrow 1)^j a_j / j!, j = 0, 1, ..., n$.

To simplify the notation, we introduce $\hat{f}_n(s) = s^{-(n+1)} f_n(s)$, which appears quite often in the proof of Theorem 2. Moreover, we have the following lemma, cf. Lemma 2 in [13].

Lemma 2 *If $\phi_{n-1} < \infty$ ($n \in \mathbf{N}$), then for s increasing*
(i) $\hat{f}_{n-1}(s)$ is decreasing;
(ii) $s\hat{f}_{n-1}(s)$ is increasing.

The following lemma (cf. Lemma 2.2 in [8]), which is an extension of Theorem 8.1.6 in [3], links the regularly varying tail behavior of $\Pr\{X > t\}$ for $t \to \infty$ to the behavior of its L-S transform $f(s)$. It plays a key role in the proof of our main results.

Lemma 3 *Let X be a random variable with L-S transform $f(s)$, $L(t)$ a slowly varying function, $\nu \in (n, n+1)$ ($n \in \mathbf{N}$) and $C \geq 0$. Then the following are equivalent:*
(i) $\Pr\{X > t\} = [C + \mathrm{o}(1)]L(t)/t^{\nu}, \;\; t \to \infty.$
(ii) $E\{X^n\} < \infty$ and $f_n(s) = (\Leftrightarrow 1)^n, \; (1 \Leftrightarrow \nu)[C + \mathrm{o}(1)]L(1/s)s^{\nu}, \;\; s \downarrow 0$.

The next lemma links the assumptions on the service time distribution functions made in Section 2 and their corresponding L-S transform functions.

Lemma 4 *(i) Assumption (i) in (7) implies that, as* $s \downarrow 0$,

$$\beta_1\{s\} = 1 \Leftrightarrow \beta_1 s + (\beta_1 s)^\nu L(1/\beta_1 s) + \mathrm{o}\Big((\beta_1 s)^\nu L(1/\beta_1 s)\Big), \tag{9}$$

$$\beta_2\{s\} = 1 \Leftrightarrow \beta_2 s + \mathrm{o}\Big((\beta_1 s)^\nu L(1/\beta_1 s)\Big); \tag{10}$$

(ii) Assumption (ii) in (7) implies that, as $s \downarrow 0$,

$$\beta_1\{s\} = 1 \Leftrightarrow \beta_1 s + \mathrm{o}\Big((\beta_1 s)^\mu\Big) \text{ where } \nu < \mu < 2, \tag{11}$$

$$\beta_2\{s\} = 1 \Leftrightarrow \beta_2 s + (\beta_2 s)^\nu L(1/\beta_2 s) + \mathrm{o}\Big((\beta_2 s)^\nu L(1/\beta_2 s)\Big); \tag{12}$$

(iii) Assumption (iii) in (7) implies that, as $s \downarrow 0$,

$$\beta_1\{s\} = 1 \Leftrightarrow \beta_1 s + (\beta_1 s)^\nu L(1/\beta_1 s) + \mathrm{o}\Big((\beta_1 s)^\nu L(1/\beta_1 s)\Big), \tag{13}$$

$$\beta_2\{s\} = 1 \Leftrightarrow \beta_2 s + \alpha(\beta_2 s)^\nu L(1/\beta_2 s) + \mathrm{o}\Big((\beta_2 s)^\nu L(1/\beta_2 s)\Big); \tag{14}$$

(iv) Assumption (iv) in (7) implies that, as $s \downarrow 0$,

$$\beta_1\{s\} = 1 \Leftrightarrow \beta_1 s + (\beta_1 s)^\nu L_1(1/\beta_1 s) + \mathrm{o}\Big((\beta_1 s)^\nu L_1(1/\beta_1 s)\Big), \tag{15}$$

$$\beta_2\{s\} = 1 \Leftrightarrow \beta_2 s + (\beta_2 s)^\nu L_2(1/\beta_2 s) + \mathrm{o}\Big((\beta_2 s)^\nu L_2(1/\beta_2 s)\Big), \tag{16}$$

where $lim_{t\to\infty} L_1(t)/L_2(t) = 0$.

Proof. We only prove (i), the proof for the rest will be similar. Equality (9) immediately follows from (i) in (7), by using Theorem 8.1.6 in [3]. Since

$$\int_0^\infty t^\mu \mathrm{d}B_2(t) < \infty,$$

it follows that

$$1 \Leftrightarrow B_2(t) = \mathrm{o}\left(\left(\frac{t}{\beta_2}\right)^{-\mu}\right).$$

Applying Lemma 1 to the above equality, we have for $s \downarrow 0$,

$$\beta_2\{s\} = 1 \Leftrightarrow \beta_2 s + \mathrm{o}\Big((\beta_2 s)^\mu\Big) \text{ where } \mu > \nu.$$

Moreover, it follows from Proposition 1.3.6 (v) in [3] that $L(1/s) = \mathrm{o}(s^{\nu-\mu})$. Hence (10) follows. □

The next lemma characterizes a property of slowly varying functions. It will be used in Section 4.

Lemma 5 *Let $L(x)$ be a slowly varying function, $t(x)$ be a positive function such that* $\lim_{x\to\infty} t(x)/x =$ *a where $0 < a < \infty$. Then for a constant ν ($\nu \in \mathbf{R}$),*

$$\lim_{x\to\infty} \frac{\{t(x)\}^{\nu} L(t(x))}{x^{\nu} L(x)} = a^{\nu}.$$

Proof. We only need to prove

$$\lim_{x\to\infty} \frac{L(t(x))}{L(x)} = 1.$$

Define:

$$\lambda(x) := \frac{t(x)}{x},$$

thus $\lambda(x) \in [a/2, 2a]$ for sufficiently large x. Applying the Uniform Convergence Theorem, cf. Theorem 1.2.1 in [3], we obtain

$$\lim_{x\to\infty} \frac{L(\lambda(x)x)}{L(x)} = 1,$$

and the result follows. □

4 The class-2 waiting time distribution

Denote by $P_1(t)$ the busy period distribution in an $M/G/1$ queue with only class-1 customers and by $\mu_1\{s\}$ the L-S transform of $P_1(t)$. Let W_2 be a random variable with distribution the steady-state waiting time distribution $W_2(t)$ of class-2 customers, and $w_2\{s\}$ the L-S transform of $W_2(t)$ where $s \geq 0$. In this section we introduce the explicit expression for $w_2\{s\}$ and its asymptotic properties as $s \downarrow 0$, when one of the assumptions in Section 2 is satisfied. From (2.14) in [1] we have

$$w_2\{s\} = \frac{1 \Leftrightarrow \rho}{1 \Leftrightarrow \rho f(s)}, \tag{17}$$

where

$$f(s) := \frac{\rho_1}{\rho_1 + \rho_2} h_0^{(1)}(s) + \frac{\rho_2}{\rho_1 + \rho_2} \beta_{2e}(z(s)), \tag{18}$$

$$h_0^{(1)}(s) := \frac{1 \Leftrightarrow \mu_1\{s\}}{\beta_1 s + \rho_1 \Leftrightarrow \rho_1 \mu_1\{s\}}, \tag{19}$$

$$z := z(s) = s + \lambda_1 \Leftrightarrow \lambda_1 \mu_1\{s\}, \tag{20}$$

for $\beta_{2e}\{s\}$ in (6). Note that there are minor differences between the above formula and the formula which was obtained by Abate and Whitt in [1] caused by their choice of $\beta_1 = 1$. Denote by $F_2(t)$ the probability distribution function with L-S transform $f_2(s) := \beta_{2e}\{z\}$.

As explained in [1], $h_0^{(1)}(s)$ is the L-S transform of the high-priority server-occupancy distribution function $H_0^{(1)}(t)$, which is defined by

$$H_0^{(1)}(t) = (1 \Leftrightarrow P_{00}^{(1)}(t))/\rho,$$

where $P_{00}^{(1)}(t)$ is the high-priority emptiness probability, i.e., the probability that the system has no class-1 customers at time t given that it had none at time 0. Actually an expression for $w_2\{s\}$ has been known for a long time, cf. Section III.3.6 of [11], but the representation in (17), which is similar to the Pollaczek-Khintchine form for the ordinary $M/G/1$ waiting time transform, appears to be new and is a suitable starting point for our analysis.

For the sake of simplicity, let us make the convention that $\beta_{j,n}(s)$, $\beta_{je,n}(s)$, $\mu_{j,n}(s)$, $h_{0,n}^{(1)}(s)$ and $f_{2,n}(s)$ stand for the function defined in (8) with $f(s)$ replaced by $\beta_j\{s\}$, $\beta_{je}\{s\}$, $\mu_j\{s\}$, $h_0^{(1)}(s)$ and $f_2(s)$ respectively.

The following lemma establishes a relation among $\beta_1\{s\}$, $\beta_{1e}\{s\}$, $\mu_1\{s\}$ and $h_0^{(1)}(s)$.

Lemma 6 *For $n < \nu < n+1$ ($n \in \mathbf{N}$), $C \geq 0$, the following statements are equivalent,*

(i) $\beta_{1,n}(s) = [C + \mathrm{o}(1)](\Leftrightarrow 1)^n, (1 \Leftrightarrow \nu)(\beta_1 s)^\nu L(1/\beta_1 s)$ *for* $s \downarrow 0$*;*

(ii) $\beta_{1e,n-1}(s) = [C + \mathrm{o}(1)](\Leftrightarrow 1)^{n-1}, (1 \Leftrightarrow \nu)(\beta_1 s)^{\nu-1} L(1/\beta_1 s)$ *for* $s \downarrow 0$*;*

(iii) $\mu_{1,n}(s) = [C + \mathrm{o}(1)](\Leftrightarrow 1)^n \frac{\Gamma(1-\nu)}{1-\rho_1}\left(\frac{\beta_1 s}{1-\rho_1}\right)^\nu L(1/\beta_1 s)$ *for* $s \downarrow 0$*;*

(iv) $h_{0,n-1}^{(1)}(s) = [C + \mathrm{o}(1)](\Leftrightarrow 1)^{n-1}, (1 \Leftrightarrow \nu)\left(\frac{\beta_1 s}{1-\rho_1}\right)^{\nu-1} L(1/\beta_1 s)$ *for* $s \downarrow 0$*.*

Proof. $(i) \Leftrightarrow (ii)$ follows from (6) immediately.
$(i) \Leftrightarrow (iii)$. By (20) we have

$$\lim_{s\downarrow 0} \frac{z(s)}{s} = \frac{1}{1 \Leftrightarrow \rho_1}. \tag{21}$$

Corollary 1 in [13] states that if the n-th moment of $B_1(t)$ or the n-th moment of $P_1(t)$ exists, then

$$(1 \Leftrightarrow \rho_1)\mu_{1,n}(s) = \beta_{1,n}(z) + \mathrm{O}\left((\beta_1 z)^{n+1}\right), \text{ for } s \downarrow 0. \tag{22}$$

Combine (21) and (22) to yield that (i) and (ii) are equivalent.
$(ii) \Leftrightarrow (iv)$. First we shall show that for $s \downarrow 0$,

$$h_{0,n-1}^{(1)}(s) \Leftrightarrow \beta_{1e,n-1}(z) = \mathrm{O}\left((\beta_1 s)^n\right). \tag{23}$$

Since $\mu_1\{s\} = \beta_1\{z\}$, we have

$$\begin{aligned} h_0^{(1)}(s) &= \frac{1 \Leftrightarrow \mu_1\{s\}}{\beta_1 s + \rho_1 \Leftrightarrow \rho_1 \mu_1\{s\}} \\ &= \frac{1 \Leftrightarrow \beta_1\{z\}}{\beta_1 z} \\ &= \beta_{1e}\{z\}. \end{aligned}$$

Applying Lemma 9 below with $g_1(s)$ and $g_2(z)$ replaced by $h_0^{(1)}(s)$ and $\beta_{1e}\{z\}$ respectively, (23) follows. Next using a similar argument as in the proof of $(i) \Leftrightarrow (iii)$ yields the result. □

By the above lemma, we may deduce the asymptotic properties of $\mu_1\{s\}$ and $\beta_{2e}\{s + \lambda_1 \Leftrightarrow \lambda_1 \mu_1\{s\}\}$ for $s \downarrow 0$, which appear in the expression (17) of $w_2\{s\}$ and determine the asymptotic behavior of $w_2\{s\}$ for $s \downarrow 0$ completely.

Lemma 7 *(i) If assumption (i) in (7) holds, then as $s \downarrow 0$,*

$$\mu_1\{s\} = 1 \Leftrightarrow \frac{\beta_1 s}{1 \Leftrightarrow \rho_1} + \frac{(\beta_1 s)^\nu L(1/\beta_1 s)}{(1 \Leftrightarrow \rho_1)^{\nu+1}} + \mathrm{o}\Big((\beta_1 s)^\nu L(1/\beta_1 s)\Big), \tag{24}$$

$$\beta_{2e}\{s + \lambda_1 \Leftrightarrow \lambda_1 \mu_1\{s\}\} = 1 + \mathrm{o}\Big((\beta_1 s)^{\nu-1} L(1/\beta_1 s)\Big); \tag{25}$$

(ii) If assumption (ii) or (iv) in (7) holds, then as $s \downarrow 0$,

$$\mu_1\{s\} = 1 \Leftrightarrow \frac{\beta_1 s}{1 \Leftrightarrow \rho_1} + \mathrm{o}\Big((\beta_1 s)^\nu L(1/\beta_1 s)\Big), \tag{26}$$

$$\beta_{2e}\{s + \lambda_1 \Leftrightarrow \lambda_1 \mu_1\{s\}\} = 1 \Leftrightarrow \left(\frac{\beta_2 s}{1 \Leftrightarrow \rho_1}\right)^{\nu-1} L(1/\beta_1 s) + \mathrm{o}\Big((\beta_2 s)^{\nu-1} L(1/\beta_1 s)\Big); \tag{27}$$

(iii) If assumption (iii) in (7) holds, then as $s \downarrow 0$,

$$\mu_1\{s\} = 1 \Leftrightarrow \frac{\beta_1 s}{1 \Leftrightarrow \rho_1} + (\beta_1 s)^\nu L(1/\beta_1 s) + \mathrm{o}\Big((\beta_1 s)^\nu L(1/\beta_1 s)\Big), \tag{28}$$

$$\beta_{2e}\{s + \lambda_1 \Leftrightarrow \lambda_1 \mu_1\{s\}\} = 1 \Leftrightarrow \alpha \left(\frac{\beta_2 s}{1 \Leftrightarrow \rho_1}\right)^{\nu-1} L(1/\beta_1 s) + \mathrm{o}\Big((\beta_2 s)^{\nu-1} L(1/\beta_1 s)\Big). \tag{29}$$

Proof. (i) Since (i) in (7) holds, it follows from the main theorem in [13] that

$$1 \Leftrightarrow P_1(t) \sim \Leftrightarrow \frac{1}{,\ (1 \Leftrightarrow \nu)(1 \Leftrightarrow \rho_1)^{1+\nu}} (t/\beta_1)^{-\nu} L(t/\beta_1) \text{ as } t \to \infty, \tag{30}$$

where $P_1(t)$ is the class-1 busy-period distribution function. Using Lemma 3, (30) leads to (24) immediately. From (20) and (24), we have for $s \downarrow 0$,

$$\frac{z(s)}{s} = \frac{1}{1 \Leftrightarrow \rho_1} + \mathrm{o}\Big((\beta_1 s)^{\nu-1} L(1/\beta_1 s)\Big).$$

By Lemma 4 (i) and Lemma 6, we have

$$\beta_{2e}\{s\} = 1 \Leftrightarrow \mathrm{o}\Big((\beta_2 s)^{\nu-1} L(1/\beta_2 s)\Big) \text{ as } s \downarrow 0. \tag{31}$$

We write

$$\frac{1 \Leftrightarrow \beta_{2e}\{z(s)\}}{(\beta_2 s)^{\nu-1} L(1/\beta_2 s)} = \frac{1 \Leftrightarrow \beta_{2e}(z(s))}{\{z(s)\}^{\nu-1} L(1/z(s))} \frac{\{z(s)\}^{\nu-1} L(1/z(s))}{(\beta_2 s)^{\nu-1} L(1/\beta_2 s)}. \tag{32}$$

Taking the limit on the two sides of the above equality for $s \downarrow 0$, and applying Lemma 5 and (31), we get

$$\lim_{s \downarrow 0} \frac{1 \Leftrightarrow \beta_{2e}\{z(s)\}}{(\beta_2 s)^{\nu-1} L(1/\beta_2 s)} = 0.$$

(ii) We only show the case in which (ii) in (7) is satisfied, i.e., by Lemma 4, (11) and (12) hold. By Assumption (7) (ii),

$$\int_0^\infty t^\mu \mathrm{d}B_1(t) < \infty,$$

which implies that

$$1 \Leftrightarrow B_1(t) = \mathrm{o}((t/\beta_1)^{-\mu}) \text{ as } t \to \infty.$$

Thus, applying Lemma 3 yields

$$\beta_{1,1}\{s\} = \mathrm{o}((\beta_1 s)^{\mu}) \text{ as } s \downarrow 0,$$

subsequently applying Lemma 6 yields

$$\mu_{1,1}\{s\} = \mathrm{o}((\beta_1 s)^{\mu}),$$

which implies (26).

From (12) it follows that, for $s \downarrow 0$,

$$1 \Leftrightarrow \beta_{2e}\{s\} = (\beta_2 s)^{\nu-1} L(1/\beta_2 s) + \mathrm{o}\Big((\beta_2 s)^{\nu-1} L(1/\beta_2 s)\Big). \tag{33}$$

By the above equality and Lemma 5,

$$\lim_{s\downarrow 0} \frac{1 \Leftrightarrow \beta_{2e}\{s + \lambda_1 \Leftrightarrow \lambda_1 \mu_1\{s\}\}}{\left(\frac{\beta_2 s}{1-\rho_1}\right)^{\nu-1} L(1/\beta_2 s)} = 1.$$

(iii) By similar arguments as in the proof of (ii), (28) and (29) follow. □

To obtain our heavy-traffic limit theorem, we rewrite $w_2\{s\}$ into the following form, which plays a key role to prove the heavy-traffic limit theorem.

Lemma 8 *Let $h_0^{(1)}(s)$, $z(s)$ and $\beta_{2e}\{s\}$ be given by (19), (20) and (6) respectively.*
(i) Assumption (i) in (7) implies that $w_2\{s\}$ can be written as

$$w_2\{s\} = \left(1 + \frac{\rho_1(\beta_1 s)^{\nu-1} L(1/\beta_1 s)}{(1 \Leftrightarrow \rho)(1 \Leftrightarrow \rho_1)^{\nu-1}} + \frac{H_1(s)}{1 \Leftrightarrow \rho}\right)^{-1} \tag{34}$$

with

$$H_1(s) = \rho_1[1 \Leftrightarrow h_0^{(1)}(s)] + \rho_2[1 \Leftrightarrow \beta_{2e}\{z(s)\}] \Leftrightarrow \frac{\rho_1(\beta_1 s)^{\nu-1} L(1/\beta_1 s)}{(1 \Leftrightarrow \rho_1)^{\nu-1}}, \tag{35}$$

where

$$\lim_{s\downarrow 0} \frac{H_1(s)}{(\beta_1 s)^{\nu-1} L(1/\beta_1 s)} = 0. \tag{36}$$

(ii) Assumption (ii) or (iv) in (7) implies that $w_2\{s\}$ can be written as

$$w_2\{s\} = \left(1 + \frac{\rho_2(\beta_2 s)^{\nu-1} L(1/\beta_2 s)}{(1 \Leftrightarrow \rho)(1 \Leftrightarrow \rho_1)^{\nu-1}} + \frac{H_2(s)}{1 \Leftrightarrow \rho}\right)^{-1} \tag{37}$$

where

$$H_2(s) = \rho_1[1 \Leftrightarrow h_0^{(1)}(s)] + \rho_2[1 \Leftrightarrow \beta_{2e}\{z(s)\}] \Leftrightarrow \frac{\rho_2(\beta_2 s)^{\nu-1} L(1/\beta_2 s)}{(1 \Leftrightarrow \rho_1)^{\nu-1}} \tag{38}$$

satisfies (36) and with $H_1(s)$ being replaced by $H_2(s)$.
(iii) Assumption (iii) in (7) implies that $w_2\{s\}$ can be written as

$$w_2\{s\} = \left(1 + \frac{\rho_1(\beta_1 s)^{\nu-1} L(1/\beta_1 s)}{(1 \Leftrightarrow \rho)(1 \Leftrightarrow \rho_1)^{\nu-1}} + \frac{\alpha\rho_2(\beta_2 s)^{\nu-1} L(1/\beta_1 s)}{(1 \Leftrightarrow \rho)(1 \Leftrightarrow \rho_1)^{\nu-1}} + \frac{H_3(s)}{1 \Leftrightarrow \rho}\right)^{-1}, \tag{39}$$

where

$$H_3(s) = \rho_1[1 \Leftrightarrow h_0^{(1)}(s)] + \rho_2[1 \Leftrightarrow \beta_{2e}\{z(s)\}] \Leftrightarrow \frac{\rho_1(\beta_1 s)^{\nu-1}L(1/\beta_1 s)}{(1 \Leftrightarrow \rho_1)^{\nu-1}} \Leftrightarrow \frac{\alpha\rho_2(\beta_2 s)^{\nu-1}L(1/\beta_1 s)}{(1 \Leftrightarrow \rho_1)^{\nu-1}} \quad (40)$$

satisfies (36) and with $H_1(s)$ being replaced by $H_3(s)$.

Proof. We only prove (i). In a similar way, by using Lemma 7 we can show (ii) and (iii). By Lemma 7 (i), Equalities (24) and (25) follow. Substituting (24) into (19), we get, as $s \downarrow 0$,

$$1 \Leftrightarrow h_0^{(1)}(s) = \frac{(\beta_1 s)^{\nu-1}L(1/\beta_1 s)}{(1 \Leftrightarrow \rho_1)^{\nu-1}} + \mathrm{o}\Big((\beta_1 s)^{\nu-1}L(1/\beta_1 s)\Big). \quad (41)$$

Rewrite (17) as

$$w_2\{s\} = \left(1 + \frac{\rho_1}{1 \Leftrightarrow \rho}[1 \Leftrightarrow h_0^{(1)}(s)] + \frac{\rho_2}{1 \Leftrightarrow \rho}[1 \Leftrightarrow \beta_{2e}\{z(s)\}]\right)^{-1}. \quad (42)$$

Replacing $H_1(s)$ in (34) with the right-hand side of (35) gives (42). Dividing $H_1(s)$ by $s^{\nu-1}L(1/s)$, substituting (41) and (25) into (35), and taking the limit for $s \downarrow 0$, we obtain (36). □

Note that in Equalities (34), (37) and (39), the factor $1/(1 \Leftrightarrow \rho_1)$ does not occur in the function $H_j(s)$ for $j = 1, 2, 3$; this plays a key role in proving our heavy-traffic limit theorem. Actually Lemma 8 and Lemma 3 imply that the stationary class-2 waiting time distribution is regularly varying of index $1 \Leftrightarrow \nu$ $(1 < \nu < 2)$ if one of the assumptions in (7) holds.

Theorem 1 *If one of the assumptions in (7) holds, then the stationary class-2 waiting time distribution $W_2(t)$ is regularly varying of index $1 \Leftrightarrow \nu$, $1 < \nu < 2$, i.e.,*

$$1 \Leftrightarrow W_2(t) \sim Mt^{1-\nu}L(t), t \to \infty.$$

E.g., if assumption (i) in (7) holds, then

$$1 \Leftrightarrow W_2(t) \sim \Leftrightarrow \frac{(\nu \Leftrightarrow 1)\rho_1(t/\beta_1)^{1-\nu}L(t/\beta_1)}{,\ (1 \Leftrightarrow \nu)(1 \Leftrightarrow \rho_1)^{\nu-1}(1 \Leftrightarrow \rho)},\ t \to \infty.$$

Remark 1. One can prove similar statements as in Theorem 1 for the case $\nu \geq 2$. In fact Theorem 9.3 in [1] provides similar results for the case of a regularly varying service time distribution of the class-2 customers. But the condition we require in our theorem is weaker than that in Theorem 9.3. There it is assumed that $B_2(\cdot)$ is regularly varying with index $\nu \geq 2$, and for the L-S transform of $B_1(t)$, there exists a $s^* > 0$ such that $\beta_1\{\Leftrightarrow s^*\} = \infty$, and $\beta_1\{s\} < \infty$ for $s < s^*$, i.e., the tail behavior of the high priority class is less heavy than that of some negative exponential distribution.

5 Links between the service time distributions and the stationary class-2 waiting time distribution

As we have proved, if one of the service time distributions has a regularly varying tail with index $\Leftrightarrow \nu$, $\nu > 1$, and the other one has a less heavy tail behavior, then the stationary class-2 waiting time distribution $W_2(t)$ has a regularly varying tail with index $1 \Leftrightarrow \nu$. Conversely, if $W_2(t)$ has a regularly varying tail with index $1 \Leftrightarrow \nu$ where $\nu > 1$, and the class-1 service time distribution $B_1(t)$ has a "less heavy tail than $t^{-\nu}$", then the class-2 service time has a regularly varying tail behavior with index $\Leftrightarrow \nu$. We shall prove this in Theorem 2.

First we introduce an inverse function of $z(s)$ defined in (20):

$$s(z) := z \Leftrightarrow \lambda_1 + \lambda_1 \beta_1\{z\}. \tag{43}$$

In the following lemma, $g_1(\cdot)$ and $g_2(\cdot)$ are some arbitrary functions which later will be given specific meaning.

Lemma 9 *Assume the n-th moment of $B_1(t)$ exists and $g_1(s) \equiv g_2(z)$ where z is defined by (20), then*

(i) for $k = 1, \cdots, n \Leftrightarrow 1$, the k-th derivative of $g_1(\cdot)$ at point 0 exists if and only if the k-th derivative of $g_2(\cdot)$ at point 0 exists.

(ii) if the k-th derivative of $g_1(\cdot)$ at point 0 exists or the k-th derivative of $g_2(\cdot)$ at point 0 exists, then there exist polynomials R_k and $P_{k,m}$ ($m = 1, \cdots, k$) in s such that

$$s^{-(k+1)}\{g_{1,k}(s) \Leftrightarrow g_{2,k}(z)\} = R_k(s) + s^{-(k+1)} \sum_{m=1}^{k} \{\mu_{1,k+1}(s)\}^m P_{k,m}(s). \tag{44}$$

Moreover, if $\mu_{1,k+1}(s) = \mathrm{o}(s^{\mu})$ where $k+1 < \mu < k+2$, then

$$s^{-(k+1)}\{g_{1,k}(s) \Leftrightarrow g_{2,k}(z)\} = R_k(0) + \mathrm{o}(s^{\mu-k-1}) \text{ for } s \downarrow 0. \tag{45}$$

Proof. First we prove (i). Assume that $g_1(\cdot)$ has a k-th derivative at point 0. Hence there exists a polynomial $\sum_{j=0}^{k} g_{1j} s^j$ such that

$$g_1(s) = \sum_{j=0}^{k} g_{1j} s^j + \mathrm{o}(s^k) \text{ for } s \downarrow 0. \tag{46}$$

We may write

$$s(z) = \sum_{j=1}^{k+1} \alpha_j z^j + (\Leftrightarrow 1)^{(k+1)} \rho_1 \beta_{1,k+1}(z),$$

which follows from (43) and the fact that $B_1(t)$ has finite $(k+1)$-th moment. In (46) replace $g_1(s)$ by $g_2(z)$ and s on the right-hand side by the right-hand side of the above equation, and rearrange it to obtain

$$g_2(z) = \sum_{j=0}^{k} g_{2j} z^j + \mathrm{o}(z^k),$$

which implies the result by using Lemma 1. The proof for the converse direction is similar, by writing

$$z(s) = \sum_{j=0}^{k+1} c_j s^j + (\Leftrightarrow 1)^{k+1} \rho_1 \mu_{1,k+1}(s), \tag{47}$$

where $\mu_{1,k+1}(s)$ is such that

$$\lim_{s\downarrow 0} s^{-(k+2)}\{(1 \Leftrightarrow \rho_1)\mu_{1,k+1}(s) \Leftrightarrow \beta_{1,k+1}(z)\} = 0,$$

which follows from Corollary 1 in [13]. We omit the proof.
Next we prove (ii). Since both $g_1(\cdot)$ and $g_2(\cdot)$ have a k-th derivative at 0, we may write

$$g_{1,k}(s) \Leftrightarrow g_{2,k}(z) = (\Leftrightarrow 1)^k \left(\sum_{j=0}^{k} g_{1j} s^j \Leftrightarrow \sum_{j=0}^{k} g_{2j} z^j \right).$$

Replace z in the above equation by (47) and rearrange slightly to obtain

$$g_{1,k}(s) \Leftrightarrow g_{2,k}(z) = Q_k(s) + \sum_{i=1}^{k} \{\mu_{1,k+1}(s)\}^i P_{i,m}(s). \tag{48}$$

It follows from (8) that

$$\lim_{s\downarrow 0} s^{-k}(g_{1,k}(s) \Leftrightarrow g_{2,k}(z)) = 0. \tag{49}$$

Since $\lim_{s\downarrow 0} s^{-k}\mu_{1,k}(s) = 0$, it follows from (48) and (49) that $\lim_{s\downarrow 0} s^{-k}Q_k(s) = 0$, which implies that $R_k(s) = s^{-(k+1)}Q_k(s)$ is a polynomial. Multiplying (48) by $s^{-(k+1)}$ gives the result. From (44) we can derive (45) directly. □

The next theorem establishes a relation between the asymptotic behavior of the service time distributions and the class-2 waiting time distribution.

Theorem 2 *If $W_2(t)$ has a regularly varying tail with index $1 \Leftrightarrow \nu$, i.e.,*

$$1 \Leftrightarrow W_2(t) \sim \frac{(\nu \Leftrightarrow 1)\rho_2 (t/\beta_2)^{1-\nu} L(t/\beta_2)}{(1 \Leftrightarrow \rho_1)^{\nu-1}(1 \Leftrightarrow \rho)} \text{ for } t \to \infty, \tag{50}$$

where $\nu > 1$ and $L(t)$ is a slowly varying function, and $B_1(t)$ has a less heavy tail than $t^{-\nu}$, then

$$1 \Leftrightarrow B_2(t) \sim (t/\beta_2)^{-\nu} L(t/\beta_2) \text{ for } t \to \infty. \tag{51}$$

Proof. Let $F(t)$ be the distribution function with L-S transform $f(s)$ which is defined in (18). Obviously (18) implies that

$$F(t) = \frac{\rho_1}{\rho_1 + \rho_2} H_0^{(1)}(t) + \frac{\rho_2}{\rho_1 + \rho_2} F_2(t), \tag{52}$$

where $F_2(t)$ is the distribution function with L-S transform $f_2(s) = \beta_{2e}\{z\}$, as introduced in the first paragraph of Section 4. Applying Theorem 1 in [9], we obtain that (50) implies that

$$1 \Leftrightarrow F(t) \sim \frac{(\nu \Leftrightarrow 1)\rho_2 (t/\beta_2)^{1-\nu} L(t/\beta_2)}{(\rho_1 + \rho_2)(1 \Leftrightarrow \rho_1)^{\nu-1}} \text{ for } t \to \infty. \tag{53}$$

Since $B_1(t)$ has a less heavy tail than that of $t^{-\nu}$, it follows that there exists a noninteger $\mu > \nu$ such that

$$\int_0^\infty t^\mu \mathrm{d}B_1(t) < \infty.$$

The above relation implies that

$$1 \Leftrightarrow B_1(t) = \mathrm{o}\Big((t/\beta_1)^{-\mu}\Big) \text{ for } t \to \infty. \tag{54}$$

Applying Lemma 3 and Lemma 6 it follows from (54) that

$$1 \Leftrightarrow H_0^{(1)}(t) = \mathrm{o}\Big((t/\beta_1)^{-\mu}\Big),$$

which in combination with (53) and (52) yields that

$$1 \Leftrightarrow F_2(t) \sim \frac{(\nu \Leftrightarrow 1)(t/\beta_2)^{1-\nu} L(t/\beta_2)}{(1 \Leftrightarrow \rho_1)^{\nu-1}} \text{ for } t \to \infty. \tag{55}$$

We shall show that (51) holds first for noninteger ν and subsequently for integer ν.
(i) ν is not an integer.
Hence there exists an integer n such that $n < \nu < n+1$ where $n \geq 1$. Without loss of generality, we may assume that $\nu < \mu < n+1$. By Lemma 3 it follows from (54) that for $s \downarrow 0$,

$$\mu_{1,n}(s) = \mathrm{o}\Big((\beta_1 s)^{\mu}\Big). \tag{56}$$

We shall show that

$$\beta_{2e,n-1}(s) = (\Leftrightarrow 1)^{n-1},\ (1 \Leftrightarrow \nu)\frac{(\beta_2 s)^{\nu-1} L(1/\beta_2 s)}{(1 \Leftrightarrow \rho_1)^{\nu-1}} + \mathrm{o}\left((\beta_2 s)^{\nu-1} L(1/\beta_2 s)\right). \tag{57}$$

Since $f_2(s)$ is the L-S transform of $F_2(t)$, by (55) and applying Lemma 3 we obtain

$$f_{2,n-1}(s) \sim (\Leftrightarrow 1)^{n-1},\ (1 \Leftrightarrow \nu)\frac{(\beta_2 s)^{\nu-1} L(1/\beta_2 s)}{(1 \Leftrightarrow \rho_1)^{\nu-1}}. \tag{58}$$

Because $f_2(s) = \beta_{2e}\{z\}$, applying Lemma 9 leads to (45) with $g_{1,n-1}(s)$ and $g_{2,n-1}(z)$ replaced by $f_{2,n-1}(s)$ and $\beta_{2e,n-1}(z)$, i.e.,

$$f_{2,n-1}(s) = \beta_{2e,n-1}(z) + \mathrm{O}\Big((\beta_2 s)^n\Big).$$

Dividing by $(\beta_2 z)^{\nu-1} L(1/\beta_2 z)$ on both sides of the above equation and noting that

$$\lim_{z \downarrow 0} \frac{s^{\nu-1} L(1/s)}{z^{\nu-1} L(1/z)} = (1 \Leftrightarrow \rho_1)^{\nu-1},$$

it follows from (58) that for $z \downarrow 0$,

$$\beta_{2e,n-1}(z) \sim (\Leftrightarrow 1)^{n-1},\ (1 \Leftrightarrow \nu)(\beta_2 z)^{\nu-1} L(1/\beta_2 z),$$

which implies that (51) holds, by the equivalence of (i) and (ii) in Lemma 6.
(ii) ν is an integer, viz., $\nu = 2, 3, \cdots$.
Firstly we prove the case that $\nu \geq 3$. Recall that $\hat{g}_n(s)$ denotes $s^{-(n+1)} g_n(s)$. As proved, $1 \Leftrightarrow F_2(t) \sim (\nu \Leftrightarrow 1)(t/\beta_2)^{1-\nu} L(t/\beta_2)/(1 \Leftrightarrow \rho_1)^{\nu-1}$ where $\nu \in \{3, 4, \cdots\}$, or equivalently, by de Haan's Theorem (cf. Theorem 3.7.3 in [3]) for $x > 1$,

$$\lim_{s \downarrow 0} [a(s)]^{-1} (\hat{f}_{2,\nu-2}(s) \Leftrightarrow \hat{f}_{2,\nu-2}(xs)) = \log x, \tag{59}$$

where we can take $a(s) = L(1/\beta_2 s)/\{(1\Leftrightarrow\rho_1)^{\nu-2}(\nu\Leftrightarrow 2)!\}$. To prove that (51) holds, it is sufficient to show that

$$\lim_{s\downarrow 0}[a(s)]^{-1}[\hat{f}_{2,\nu-2}(s) \Leftrightarrow (1 \Leftrightarrow \rho_1)^{1-\nu}\hat{\beta}_{2e,\nu-2}(\frac{s}{1 \Leftrightarrow \rho_1}) \Leftrightarrow \theta_{\nu-2}] = 0, \tag{60}$$

for some constant $\theta_{\nu-2}$. If (60) holds, we may write for $x > 1$,

$$\lim_{s\downarrow 0}[a(s)]^{-1}[\hat{f}_{2,\nu-2}(xs) \Leftrightarrow (1 \Leftrightarrow \rho_1)^{1-\nu}\hat{\beta}_{2e,\nu-2}(\frac{xs}{1 \Leftrightarrow \rho_1}) \Leftrightarrow \theta_{\nu-2}] = 0. \tag{61}$$

Subtracting (60) by (61) and using (59) yields

$$\lim_{s\downarrow 0}[a(s)(1 \Leftrightarrow \rho_1)^{\nu-1}]^{-1}[\beta_{2e,\nu-2}(s) \Leftrightarrow \beta_{2e,\nu-2}(xs)] = \log x.$$

Applying the reverse statement of de Haan's Theorem (cf. Theorem 3.7.3 in [3]) to the above relation leads to (51).

To prove (60) we use the expression (44) for $k = \nu \Leftrightarrow 2$; the right-hand side of (44) will be abbreviated by $A_{\nu-2}(s)$ with $g_1(s)$ replaced by $f_2(s)$ and $g_2(z)$ replaced by $\beta_{2e}\{z\}$. Hence

$$\hat{f}_{2,\nu-2}(s) = \hat{\beta}_{2e,\nu-2}(z)(z/s)^{\nu-1} + A_{\nu-2}(s). \tag{62}$$

This suggests that we might take $\theta_{\nu-2} = A_{\nu-2}(0)$. So define

$$J_{\nu-2}(s) = [a(s)]^{-1}[\hat{f}_{2,\nu-2}(s) \Leftrightarrow (1 \Leftrightarrow \rho_1)^{1-\nu}\hat{\beta}_{2e,\nu-2}(s/(1 \Leftrightarrow \rho_1)) \Leftrightarrow A^{\nu-2}(0)]. \tag{63}$$

We shall show that $\lim_{s\downarrow 0} J_{\nu-2}(s) = 0$ in the same way as De Meyer and Teugels [13], p. 810/811. Since $\mu_{1,0}(s)$ is decreasing by Lemma 2 (i), we have

$$\hat{\mu}_{1,0}(s) \leq \lim_{s\downarrow 0}\frac{1 \Leftrightarrow \mu_1\{s\}}{s} = \frac{\beta_1}{1 \Leftrightarrow \rho_1},$$

so that

$$z = s[1 + \lambda\hat{\mu}_{1,0}(s)] \leq \frac{s}{1 \Leftrightarrow \rho_1}.$$

Again, by Lemma 2, it follows that $\hat{\beta}_{2e,\nu-2}(s)$ is decreasing and $s\hat{\beta}_{2e,\nu-2}(s)$ is increasing, therefore

$$\hat{\beta}_{2e,\nu-2}(\frac{s}{1 \Leftrightarrow \rho_1}) \leq \hat{\beta}_{2e,\nu-2}(z) \leq \frac{s}{(1 \Leftrightarrow \rho_1)z}\hat{\beta}_{2e,\nu-2}(\frac{s}{1 \Leftrightarrow \rho_1}). \tag{64}$$

By using (62) in (63) and subsequently applying the above relation, we have

$$\begin{aligned} & J_{\nu-2}(s) \\ = \; & [a(s)]^{-1}\{\hat{\beta}_{2e,\nu-2}(z)(z/s)^{1-\nu} \Leftrightarrow (1 \Leftrightarrow \rho_1)^{1-\nu}\hat{\beta}_{2e,\nu-2}(\frac{s}{1 \Leftrightarrow \rho_1}) + [A_{\nu-2}(s) \Leftrightarrow A_{\nu-2}(0)]\} \\ \geq \; & [a(s)]^{-1}\{\hat{\beta}_{2e,\nu-2}(\frac{s}{1 \Leftrightarrow \rho_1})[(z/s)^{\nu-1} \Leftrightarrow (1 \Leftrightarrow \rho_1)^{1-\nu}] + [A_{\nu-2}(s) \Leftrightarrow A_{\nu-2}(0)]\}, \end{aligned} \tag{65}$$

and

$$J_{\nu-2}(s) \leq [a(s)]^{-1}\{(1\Leftrightarrow\rho_1)^{-1}\hat{\beta}_{2e,\nu-2}(\frac{s}{1 \Leftrightarrow \rho_1})[(z/s)^{\nu-2}\Leftrightarrow(1\Leftrightarrow\rho_1)^{2-\nu}] + [A_{\nu-2}(s)\Leftrightarrow A_{\nu-2}(0)]\}. \tag{66}$$

By (20) it follows that

$$\frac{z(s)}{s} = \frac{1}{1 \Leftrightarrow \rho_1} + (\beta_1 s)\mathrm{O}(1),$$

thus

$$(z/s)^{\nu-1} \Leftrightarrow (1 \Leftrightarrow \rho_1)^{1-\nu} = (\nu \Leftrightarrow 1)(\beta_1 s)\mathrm{O}(1). \tag{67}$$

Moreover, it follows from the definition of $A_{\nu-2}(s)$ and (45) that

$$\lim_{s \downarrow 0} [a(s)]^{-1}[A_{\nu-2}(s) \Leftrightarrow A_{\nu-2}(0)] = 0. \tag{68}$$

Multiplying (59) by s, it follows that

$$\lim_{s \downarrow 0} [a(s)]^{-1}[s\hat{f}_{2,\nu-2}(s) \Leftrightarrow s\hat{f}_{2,\nu-2}(xs)] = 0,$$

on the other hand,

$$\lim_{s \downarrow 0} [a(s)]^{-1}[s\hat{f}_{2,\nu-2}(s) \Leftrightarrow s\hat{f}_{2,\nu-2}(xs)] = (1 \Leftrightarrow 1/x) \lim_{s \downarrow 0} [a(s)]^{-1} s\hat{f}_{2,\nu-2}(s).$$

The above two relations imply that $\lim_{s\downarrow 0}[a(s)]^{-1}s\hat{f}_{2,\nu-2}(s) = 0$. Consequently, it follows from (62) that $\lim_{s\downarrow 0}[a(s)]^{-1}s\beta_{2e,\nu-2}(z) = 0$, or equivalently,

$$\lim_{z \downarrow 0} [a(z)]^{-1} z\beta_{2e,\nu-2}(z) = 0. \tag{69}$$

Combining (67), (68) and (69), we have

$$\lim_{s\downarrow 0}[a(s)]^{-1}\{\hat{\beta}_{2e,\nu-2}(\tfrac{s}{1-\rho_1})[(z/s)^{\nu-1} \Leftrightarrow (1 \Leftrightarrow \rho_1)^{1-\nu}] + [A_{\nu-2}(s) \Leftrightarrow A_{\nu-2}(0)]\} = 0, \tag{70}$$

$$\lim_{s\downarrow 0}[a(s)]^{-1}\{(1 \Leftrightarrow \rho_1)^{-1}\hat{\beta}_{2e,\nu-2}(\tfrac{s}{1-\rho_1})[(z/s)^{\nu-2} \Leftrightarrow (1 \Leftrightarrow \rho_1)^{2-\nu}] + [A_{\nu-2}(s) \Leftrightarrow A_{\nu-2}(0)]\} = 0. \tag{71}$$

Therefore, combining (65), (66), (70) and (71) yields that $\lim_{s\downarrow 0} J_{\nu-2}(s) = 0$.

Secondly we prove the case that $\nu = 2$. Again we intend to show that (60) holds by taking $\theta_{\nu-2} = 0$. We define

$$J_0(s) := [a(s)]^{-1}[\hat{f}_{2,0}(s) \Leftrightarrow \frac{1}{1 \Leftrightarrow \rho_1}\hat{\beta}_{2e,0}(\frac{s}{1 \Leftrightarrow \rho_1})].$$

By the fact that $\hat{f}_{2,0}(s) = \hat{\beta}_{2e,0}(z)(z/s)$ and (64), it follows that

$$[a(s)]^{-1}\{\hat{\beta}_{2e,0}(\frac{s}{1 \Leftrightarrow \rho_1})(\frac{z}{s} \Leftrightarrow \frac{1}{1 \Leftrightarrow \rho_1})\} \le J_0(s) \le 0.$$

It follows from (67) and (69) that

$$\lim_{s \downarrow 0} [a(s)]^{-1}\{\hat{\beta}_{2e,0}(\frac{s}{1 \Leftrightarrow \rho_1})(\frac{z}{s} \Leftrightarrow \frac{1}{1 \Leftrightarrow \rho_1})\} = 0.$$

The above two relations lead to $\lim_{s\downarrow 0} J_0(s) = 0$, which implies that (60) is satisfied for $\nu = 2$ and thus (51) follows. $\square$

6 A heavy-traffic limit theorem for the queueing model with two priority classes

In [4] Boxma and Cohen have obtained heavy-traffic limit theorems for the $GI/G/1$ queue. In one of their cases they assume that the tail of the service time distribution is regularly varying with index $\Leftrightarrow\nu$ $(1 < \nu < 2)$ and the tail of the interarrival time distribution is less heavy than that of the service time distribution. Their theorem then states that the 'contracted' waiting time $\Delta(\rho)W/\beta$ converges in distribution for $\rho \uparrow 1$ to a limiting distribution $R_{\nu-1}(t)$. This distribution is specified by

$$\int_0^\infty e^{-st}\mathrm{d}R_{\nu-1}(t) = \frac{1}{1+s^{\nu-1}},$$

and the 'coefficient of contraction' $\Delta(\rho)$ is the only solution to a 'contraction equation' with the property that $\Delta(\rho) \downarrow 0$ for $\rho \uparrow 1$.

In this section we apply a similar method as in [4] to derive a heavy-traffic limit theorem for the low priority waiting time of the queueing model with two types of customers. We assume that $\rho_2 \uparrow 1 \Leftrightarrow \rho_1,\ 0 < \rho_1 < 1$ and that one of the assumptions in (7) is satisfied.

Consider the contraction equation

$$\frac{Kx^{\nu-1}L(1/x)}{1 \Leftrightarrow \rho} = 1,\ x > 0, \tag{72}$$

where K is a function of both ρ_1 and ρ_2 such that $K > c$ for some positive constant c, $L(x)$ is a slowly varying function, and denote by $\Delta(\rho_2)$ the unique root of (72) such that

$$\Delta(\rho_2) \downarrow 0 \text{ for } \rho_2 \uparrow 1 \Leftrightarrow \rho_1, \tag{73}$$

cf. [4].

We say that the solution $\Delta(\rho)$ to the contraction equation is the unique solution with the property that $\Delta(\rho) \downarrow 0$ for $\rho \uparrow 1$, if for two solutions to the contraction equation $\Delta_j(\rho)$ $(j = 1, 2)$ such that $\Delta_j(\rho) \downarrow 0$ for $\rho \uparrow 1$, the limit of the proportion of the solutions for $\rho \uparrow 1$ is equal to 1, i.e.,

$$\lim_{\rho\uparrow 1} \frac{\Delta_1(\rho)}{\Delta_2(\rho)} = 1.$$

In the following we provide a lemma which characterizes the property of the solution to the contraction equation (72).

Lemma 10 *If $L(t)$ is continuous, then there exists a unique solution $\Delta(\rho_2)$ to the contraction equation (72) with the property that $\Delta(\rho_2) \downarrow 0$ for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$.*

Proof. Since

$$\lim_{s\downarrow 0} s^{\nu-1}L(1/s) = 0,$$

by the continuity of $L(1/s)$, it follows that, for sufficiently large ρ_2 where $\rho_2 < 1 \Leftrightarrow \rho_1$, there exists at least one solution $\xi(\rho_2)$ to the equation

$$Kx^{\nu-1}L(1/x) = 1 \Leftrightarrow \rho. \tag{74}$$

Put

$$\Delta(\rho_2) = \inf\{\xi(\rho_2) : K\xi(\rho_2)^{\nu-1}L(1/\xi(\rho_2)) = 1 \Leftrightarrow \rho\}.$$

By continuity of $L(1/x)$, $\Delta(\rho_2)$ is also a solution to Equation (74). Next we show that $\Delta(\rho_2) \downarrow 0$ for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$. Assume, to the contrary, that there exists a sequence $\{\rho_{2n}\}$ $(n = 1, 2, ...)$ which tends to $1 \Leftrightarrow \rho_1$ such that, for all n, $\Delta(\rho_{2n}) > \epsilon$ for some positive constant ϵ. If n is large enough, then $1 \Leftrightarrow \rho_1 \Leftrightarrow \rho_{2n}$ is arbitrarily small. Thus for sufficiently large n, there exists at least one solution $\xi(\rho_{2n})$ to Equation (74) such that $\xi(\rho_{2n}) < \epsilon$. On the other hand, by the definition of $\Delta(\rho_2)$,

$$\xi(\rho_{2n}) \geq \Delta(\rho_{2n}) > \epsilon,$$

which is a contradiction to $\xi(\rho_{2n}) < \epsilon$. Hence

$$\lim_{\rho_2 \uparrow 1-\rho_1} \Delta(\rho_2) = 0.$$

Now we shall prove the uniqueness of the solution $\Delta(\rho_2)$ to Equation (74) with the property that $\Delta(\rho_2) \to 0$ for $\rho_2 \to 1 \Leftrightarrow \rho_1$. Let $\Delta_j(\rho_2)$ be solutions to Equation (74) with the property that $\Delta_j(\rho_2) \to 0$ for $\rho_2 \to 1 \Leftrightarrow \rho_1$, $j = 1, 2$. It is sufficient to show that if

$$\lim_{n\to\infty} \rho_{2n} = 1 \Leftrightarrow \rho_1,$$

$$\lim_{n\to\infty} \Delta_j(\rho_{2n}) = 0, \text{ for } j = 1, 2,$$

$$a = \lim_{n\to\infty} \frac{\Delta_1(\rho_{2n})}{\Delta_2(\rho_{2n})} \text{ where } 0 \leq a \leq \infty,$$

then

$$a = 1.$$

Since $\Delta_j(\rho_2)$ $(j = 1, 2)$ are solutions to Equation (74), we can write

$$K\Delta_j(\rho_2)^{\nu-1} L(1/\Delta_j(\rho_2)) = 1 \Leftrightarrow \rho.$$

It follows that

$$\frac{\Delta_1(\rho_2)^{\nu-1} L(1/\Delta_1(\rho_2))}{\Delta_2(\rho_2)^{\nu-1} L(1/\Delta_2(\rho_2))} = 1. \tag{75}$$

If $0 < a < \infty$, it follows from Lemma 5 that

$$\lim_{\rho_2 \to 1-\rho_1} \frac{\Delta_1(\rho_2)^{\nu-1} L(1/\Delta_1(\rho_2))}{\Delta_2(\rho_2)^{\nu-1} L(1/\Delta_2(\rho_2))} = a^{\nu-1}.$$

Combining the above equality and (75) gives

$$a = 1.$$

Next we prove that $a \neq 0$. Set

$$b_n := \frac{\Delta_1(\rho_{2n})}{\Delta_2(\rho_{2n})}.$$

Assume $\lim_{n\to\infty} b_n = 0$. Choose ϵ positive and small enough. Applying Theorem 1.5.2 in [3], we have

$$\frac{\Delta_1(\rho_{2n})^{\nu-1} L(1/b_n \Delta_2(\rho_{2n}))}{\Delta_2(\rho_{2n})^{\nu-1} L(1/\Delta_2(\rho_{2n}))} < b_n^{\nu-1} + \epsilon$$

for sufficiently large n, which contradicts (75). Hence $a \neq 0$. Similarly,

$$\lim_{n\to\infty} \frac{\Delta_2(\rho_{2n})}{\Delta_1(\rho_{2n})} \neq 0.$$

Consequently, $a \neq \infty$. $\square$

If explicit representations (as in [4], cf. Remark 3 below) for the service time distributions are given, i.e., the L-S transforms $\beta_j\{s\}$ $(j = 1, 2)$ of the service time distributions can be represented by (94), then one can prove that there is a unique root with the property that $\Delta(\rho_2) \downarrow 0$ for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$.

Lemma 11 *If one of the assumptions in (7) in Section 2 is satisfied, then there exists a contraction coefficient $\Delta(\rho_2)$ satisfying (72) such that $\Delta(\rho_2) \downarrow 0$ for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$, and*

$$\lim_{\rho_2 \uparrow 1-\rho_1} w_2(\Delta(\rho_2)s/\beta_1) = \frac{1}{1+s^{\nu-1}}. \tag{76}$$

Proof. We prove the case in which assumption (i) in (7) holds. Let $\Delta(\rho_2)$ be the root of the contraction function

$$\frac{\rho_1 x^{\nu-1} L(1/x)}{(1 \Leftrightarrow \rho)(1 \Leftrightarrow \rho_1)^{\nu-1}} = 1 \tag{77}$$

for which (73) holds. It follows from (73) and (77) that, for $s \geq 0$,

$$\lim_{\rho_2 \uparrow 1-\rho_1} \frac{\rho_1 (s\Delta(\rho_2))^{\nu-1} (L(1/\Delta(\rho_2)s)}{(1 \Leftrightarrow \rho)(1 \Leftrightarrow \rho_1)^{\nu-1}} = s^{\nu-1}. \tag{78}$$

By Lemma 7 (i) we have

$$w_2\{s\} = \left(1 + \frac{\rho_1(\beta_1 s)^{\nu-1} L(1/\beta_1 s)}{(1 \Leftrightarrow \rho)(1 \Leftrightarrow \rho_1)^{\nu-1}} + \frac{H_1(s)}{1 \Leftrightarrow \rho}\right)^{-1}, \tag{79}$$

where $H_1(s)$ is such that

$$\lim_{s\downarrow 0} \frac{H_1(s)}{(\beta_1 s)^{\nu-1} L(1/\beta_1 s)} = 0.$$

By (78) and the above relation,

$$\lim_{\rho_2 \uparrow 1-\rho_1} \frac{H_1(\Delta(\rho_2)s/\beta_1)}{1 \Leftrightarrow \rho} = 0, \tag{80}$$

where $H_1(s)$ is given by (35). Substituting $\Delta(\rho_2)s/\beta_1$ in $w_2\{s\}$ and taking the limit for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$ yields (76). $\square$

The analysis given above leads to the following theorem.

Theorem 3 *For the stable M/G/1 queue with two priority classes, the service time distributions $B_1(t)$ and $B_2(t)$ satisfying one of the assumptions in (7), the "contracted" waiting time*

$\Delta(\rho_2)W_2/\beta_1$ converges in distribution for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$, and the limit distribution $R_{\nu-1}(t)$ is given by: for $t \geq 0$,

$$R_{\nu-1}(t) = 1 \Leftrightarrow \sum_{n=0}^{\infty} (\Leftrightarrow 1)^n \frac{t^{n(\nu-1)}}{,\ (n(\nu \Leftrightarrow 1) + 1)}. \tag{81}$$

The coefficient of contraction $\Delta(\rho_2)$ is that root of the equation (72) with the property that $\Delta(\rho_2) \downarrow 0$ for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$, and with $K = K_1, ..., K_4$ corresponding to assumptions (i),...,(iv) in (7) respectively, where $K_1 = \frac{\rho_1}{(1-\rho_1)^{\nu-1}}$, $K_2 = \frac{\rho_2(\beta_2/\beta_1)^{\nu-1}}{(1-\rho_1)^{\nu-1}}$, $K_3 = \frac{\rho_1+\rho_2\alpha(\beta_2/\beta_1)^{\nu-1}}{(1-\rho_1)^{\nu-1}}$ and $K_4 = K_2$. Moreover, the L-S transform of $R_{\nu-1}(t)$ is

$$\int_{0-}^{\infty} e^{-st} \mathrm{d}R_{\nu-1}(t) = \frac{1}{1 + s^{\nu-1}}, \ s \geq 0. \tag{82}$$

Proof. By Lemma 11, the L-S transform of the distribution of the "contracted" waiting time $\Delta(\rho_2)W_2/\beta_1$ converges, for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$, and the limit function is given by

$$\lim_{\rho_2 \uparrow 1-\rho_1} E\{e^{-\Delta(\rho_2)W_2 s/\beta_1}\} = \lim_{\rho_2 \uparrow 1-\rho_1} w_2(\Delta(\rho_2)s/\beta_1) = \frac{1}{1 + s^{\nu-1}}. \tag{83}$$

Since $1/(1 + s^{\nu-1}) \to 1$ for $s \downarrow 0$, using the convergence theorem of Feller for L-S transforms, cf. [15], it follows that there exists a proper distribution function $R_{\nu-1}(t)$ which has L-S transform $1/(1+s^{\nu-1})$ (the Mittag-Leffler distribution). Relation (83) implies that the "contracted" waiting time distribution of $\Delta(\rho_2)W_2/\beta_1$ converges to $R_{\nu-1}(t)$ for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$. For this distribution $R_{\nu-1}(t)$ we have

$$\int_0^{\infty} e^{-st}(1 \Leftrightarrow R_{\nu-1}(t))\mathrm{d}t = \frac{s^{\nu-2}}{1 + s^{\nu-1}}, \ s \geq 0. \tag{84}$$

By applying Theorem 2 of [14] , Vol. II, p. 175, it is readily seen that: for $t \geq 0$,

$$1 \Leftrightarrow R_{\nu-1}(t) = \sum_{n=0}^{\infty} (\Leftrightarrow 1)^n \frac{t^{n(\nu-1)}}{,\ (n(\nu \Leftrightarrow 1) + 1)}, \tag{85}$$

from which we know that $R_{\nu-1}(t)$ is continuous on $[0, \infty)$. □

Remark 2. Applying Theorem 8.1.6 in [3], we obtain the tail behavior of the distribution $R_{\nu-1}(t)$:

$$1 \Leftrightarrow R_{\nu-1}(t) \sim \frac{t^{1-\nu}}{,\ (2 \Leftrightarrow \nu)} \text{ for } t \to \infty.$$

It can also be derived from the asymptotic expansion of $w_2(\Delta(\rho_2)s/\beta_1)$ for $s \downarrow 0$, and Theorem 8.1.6 in [3], that

$$\Pr\{\Delta(\rho_2)W_2/\beta_1 \geq t\} \sim \frac{t^{1-\nu}}{,\ (2 \Leftrightarrow \nu)} \text{ for } t \to \infty.$$

7 A heavy-traffic limit theorem for the queueing model with k priority classes

In this section we consider the queueing model with k priority classes where $k \geq 2$. Let the j-th priority class be indexed by j for $1 \leq j \leq k$. Denote by ρ_j the workload generated by class-j, λ_j the arrival rate of class-j, $B_j(t)$ the service time distribution of class-j, W_j the steady-state class-j waiting time for $1 \leq j \leq k$. To have a steady-state class-k waiting time distribution, we assume $\sum_{j=1}^{k} \rho_j < 1$.

Suppose one of the service time distributions has the following heavy tail behavior:

$$1 - B_i(t) \sim L(t)t^{-\nu}, \text{ as } t \to \infty, \tag{86}$$

with $L(t)$ a slowly varying function and $1 < \nu < 2$, the other service time distributions being such that, for $j \neq i$, $1 \leq j \leq k$,

$$\int_0^\infty t^{\mu_j} \mathrm{d}B_j(t) < \infty \text{ where } \mu_j > \nu,$$

or

$$1 - B_j(t) \sim L_j(t)t^{-\nu}$$

with $\lim_{t\to\infty} L_j(t)/L(t) < \infty$. Obviously only class-$k$ customers experience heavy traffic for $\rho_k \uparrow 1 - \sum_{j=1}^{k-1} \rho_j$. We can solve this problem by considering this queueing model with k priority classes as a queueing model with two priority classes. Subsequently, we use the result in Section 6 to get the heavy-traffic limit theorem for the generalized model. Let the first $k-1$ classes be the high-priority class, class-k the low-priority class in a queueing model with two priority classes. The service time distributions of the two classes in the new model are given by

$$\tilde{B}_1(t) = \frac{\sum_{j=1}^{k-1} \lambda_j B_j(t)}{\sum_{j=1}^{k-1} \lambda_j}, \tag{87}$$

$$\tilde{B}_2(t) = B_k(t). \tag{88}$$

The above assumptions imply that one of the assumptions in (7) holds for $\tilde{B}_1(t)$, $\tilde{B}_2(t)$. Hence the following heavy-traffic limit theorem holds.

Theorem 4 *For the stable $M/G/1$ queue with k ($k \geq 2$) priority classes, the above assumptions for the service time distributions $B_j(t)$, $1 \leq j \leq k$, holding, the "contracted" waiting time $\Delta(\rho_k)W_k/\beta_1$ converges in distribution for $\rho_k \uparrow 1 - \sum_{j=1}^{k-1} \rho_j$; the limit distribution $R_{\nu-1}(t)$ is given by (81), and the coefficient of contraction $\Delta(\rho_k)$ is that root of the equation (72) with the property that $\Delta(\rho_k) \downarrow 0$ for $\rho_k \uparrow 1 - \sum_{j=1}^{k-1} \rho_j$.*

8 A heavy-traffic limit theorem for the queueing model without priority

In this section we compare the low-priority waiting time W_2 in the $M/G/1$ queueing model with two priority classes and the waiting time W in the same model without priority. In the $M/G/1$ queueing model without priority, the traffic load ρ, the service time distribution $B(t)$, the L-S transform $\beta\{s\}$ of $B(t)$ and the mean β of the service time are given by

$$\rho = \rho_1 + \rho_2, \tag{89}$$

$$B(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2} B_1(t) + \frac{\lambda_2}{\lambda_1 + \lambda_2} B_2(t), \tag{90}$$

$$
\beta\{s\} = \frac{\lambda_1}{\lambda_1+\lambda_2}\beta_1\{s\} + \frac{\lambda_2}{\lambda_1+\lambda_2}\beta_2\{s\}, \tag{91}
$$

$$
\beta = \frac{\rho_1+\rho_2}{\lambda_1+\lambda_2}, \tag{92}
$$

from which it follows that

$$
\beta_e\{s\} = \frac{\rho_1\beta_{1e}\{s\}}{\rho_1+\rho_2} + \frac{\rho_2\beta_{2e}\{s\}}{\rho_1+\rho_2}. \tag{93}
$$

To get a heavy-traffic limit theorem for this model, we take $\tilde{\rho}_1 = 0$, $\tilde{\rho}_2 = \rho$, $\tilde{B}_1(t)$ is exponentially distributed and $\tilde{B}_2(t) = B(t)$. Then applying Theorem 3 yields the following heavy traffic limit theorem for the classical $M/G/1$ queue without priority discipline.

Theorem 5 *For the stable $M/G/1$ queue, the service time distribution $B(t)$ being given by (90) where $B_1(t)$ and $B_2(t)$ satisfy one of the assumptions in (7), the "contracted" waiting time $\delta(\rho_2)W/\beta_1$ converges in distribution for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$; the limit distribution $R_{\nu-1}(t)$ is given by (81), and the coefficient of contraction $\delta(\rho_2)$ is the root of the equation (72) with the property that $\delta(\rho_2) \downarrow 0$ for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$, and with $K = K_1, ..., K_4$ corresponding to assumptions (i),...,(iv) in (72) respectively, where $K_1 = \frac{\rho_1}{\rho_1+\rho_2}$, $K_2 = \frac{\rho_2(\beta_2/\beta_1)^{\nu-1}}{\rho_1+\rho_2}$, $K_3 = \frac{\rho_1+\rho_2 a(\beta_2/\beta_1)^{\nu-1}}{\rho_1+\rho_2}$ and $K_4 = K_2$.*

Remark 3. The heavy-traffic limit theorem for the steady-state waiting time in the $GI/G/1$ queue with heavy-tailed service and/or interarrival time distribution was obtained by Boxma and Cohen; see [4]. In [4] it is assumed that the L-S transform of the service time distribution $\beta\{s\}$ can be represented as: for Re $s \geq 0$,

$$
1 \Leftrightarrow \frac{1 \Leftrightarrow \beta\{s\}}{\beta s} = h(\beta s) + c_0(\beta s)^{\nu-1}L(1/\beta s), \tag{94}
$$

where

(i) $c_0 > 0$ is a constant;

(ii) $1 < \nu \leq 2$;

(iii) $h(\beta s)$ is a regular function of s for Re $s > \Leftrightarrow \delta$, $h(0) = 0$;

(iv) $L(1/\beta s)$ is regular for Re $s > 0$, and continuous for Re $s \geq 0$, except possibly at $s = 0$; $L(1/\beta s) \to b > 0$ for $|s| \to 0$, Re $s \geq 0$, with $b = \infty$ if $\nu = 2$, $\lim_{x\downarrow 0} \dfrac{L(1/\beta s x)}{L(1/\beta x)} = 1$ for Re $s \geq 0, s \neq 0$;

(v) For a $\mu \in (1,\ \nu)$:

$$
\int_0^\infty t^\mu \mathrm{d}B(t) < \infty.
$$

More generally, the L-S transform of the service time distribution can be represented as

$$
1 \Leftrightarrow \frac{1 \Leftrightarrow \beta\{s\}}{\beta s} = \sum_{i=1}^{\infty} c_i(\beta s)^{\nu_i-1}L_i(1/\beta s) + h(\beta s),
$$

where $1 < \nu_1 < \cdots < \nu_n < \cdots$, $L_i(1/s)$ satisfies (iv) in (94), c_i is a constant and $h(s)$ satisfies (iii) in (94).

Theorems 3 and 5 show that both $\Delta(\rho_2)W_2/\beta_1$ and $\delta(\rho_2)W/\beta_1$ converge to $R_{\nu-1}(t)$ in distribution for $\rho_2 \uparrow 1 \Leftrightarrow \rho_1$. The following lemma exposes the relation between $\Delta(\rho_2)$ and $\delta(\rho_2)$. We omit the proof. One can prove it by applying a similar method as in the proof of Lemma 10.

Lemma 12 *If one of the assumptions in (7) is satisfied, then*

$$\lim_{\rho_2 \uparrow 1-\rho_1} \frac{\Delta(\rho_2)}{\delta(\rho_1)} = 1 \Leftrightarrow \rho_1.$$

Apparently the result of introducing priorities is (cf. the difference in the constants K_i in Theorem 3 and 5): class-1 customers are not in heavy traffic and class-2 customers experience similar heavy traffic waiting time tail behavior as in the case without priority, apart from a scaling factor $1 \Leftrightarrow \rho_1$. ρ_1 is the fraction of time that the server is occupied by class-1 customers and $1 \Leftrightarrow \rho_1$ is the fraction of time that the server is available for class-2 customers. Actually, the following approximation seems useful:

$$1 \Leftrightarrow W_2(t) \approx 1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t), \ t \geq 0.$$

First of all, this approximation satisfies the heavy-traffic behavior indicated above. Secondly, it yields the correct mean waiting time $E(W_2) = E(W)/(1 \Leftrightarrow \rho_1)$; cf Cohen [11], Formula (II.3.64). Thirdly, it gives the correct behavior at $t = 0$ (unlike the heavy-traffic approximation). And finally, it follows from Theorem 1 (see also Remark 1) that, if one of the assumptions in (7) holds, then

$$1 \Leftrightarrow W_2(t) \sim M t^{1-\nu} L(t), \ t \to \infty. \tag{95}$$

It is well-known (cf. Cohen [9] and Pakes [18]) that, in the $M/G/1$ queue with regularly varying (even subexponential) service time distribution $B(\cdot)$,

$$1 \Leftrightarrow W(t) \sim \frac{\rho}{1 \Leftrightarrow \rho} \int_t^\infty \frac{1 \Leftrightarrow B(x)}{\beta} \mathrm{d}x, \ t \to \infty.$$

One can easily verify that this yields exactly the same tail behavior for $1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t)$ as in (95).

In fact even in the $M/G/1$ queueing model with two priority classes and the nonpreemptive discipline, only class-2 customers experience heavy traffic. This is easily seen from the following expression for $w_1\{s\}$, for $\rho_1 + \rho_2 < 1$,

$$w_1\{s\} = \frac{1 \Leftrightarrow \rho_1 \Leftrightarrow \rho_2 + \rho_2 \beta_{2e}\{s\}}{1 \Leftrightarrow \rho_1 \beta_{1e}\{s\}},$$

and for $\rho_1 + \rho_2 \geq 1$,

$$w_1\{s\} = \frac{(1 \Leftrightarrow \rho_1)\beta_{2e}\{s\}}{1 \Leftrightarrow \rho_1 \beta_{1e}\{s\}},$$

cf. Section III.3.8 in Cohen [11]. Generally, in the $M/G/1$ queueing model with k ($k \geq 2$) priority classes, nonpreemptive or preemptive resume discipline, only the lowest priority class suffers heavy traffic.

9 Applications of the heavy-traffic theorem

Theorem 3, the heavy-traffic theorem, suggests the following heavy-traffic approximation for the stationary class-2 waiting time distribution: for $0 < 1 \Leftrightarrow \rho_1 \Leftrightarrow \rho_2 << 1 \Leftrightarrow \rho_1$, and with $\Delta(\rho_2)$ specified by the contraction equation (72),

$$\Pr\{\frac{\Delta(\rho_2)W_2}{\beta_1} > t\} \approx 1 \Leftrightarrow R_{\nu-1}(t),\ t > 0, \tag{96}$$

or equivalently,

$$1 \Leftrightarrow W_2(t) = \Pr\{W_2 > t\} \approx 1 \Leftrightarrow R_{\nu-1}(\Delta(\rho_2)t/\beta_1),\ t > 0. \tag{97}$$

According to the heavy-traffic theorem, this approximation should perform very well when ρ is sufficiently close to 1. In this section we investigate whether this approximation is still useful when ρ is not very close to 1. We follow a similar procedure as [6], where such a heavy-traffic approximation for the waiting time distribution of the $M/G/1$ case without priorities is numerically investigated. We suppose that the heavy-tailed service time distributions are of the following form,

$$B_j(t) = 1 \Leftrightarrow \frac{1}{,\ (2 \Leftrightarrow \nu_j)} \int_0^\infty e^{-\theta} \frac{\theta}{(\theta + t)^{\nu_j}} \mathrm{d}\theta,\ j = 1, 2, \tag{98}$$

with $1 < \nu_j < 2$ and $,\ (\cdot)$ is the Gamma function. Note that

$$B_j(0+) = 0,$$

$$\beta_j = \int_0^\infty t \mathrm{d}B_j(t) = \frac{2 \Leftrightarrow \nu_j}{\nu_j \Leftrightarrow 1};$$

the second moment of $B_j(t)$ is infinite. As shown in [6], the L-S transform of $B_j(t)$ as given in (98) is characterized by: for Re $s \geq 0$,

$$\frac{1 \Leftrightarrow \beta_j\{s\}}{\beta_j s} = \frac{\omega}{\omega \Leftrightarrow 1}\left[1 \Leftrightarrow \frac{1}{2 \Leftrightarrow \nu_j} \frac{\omega^{2-\nu_j} \Leftrightarrow 1}{\omega \Leftrightarrow 1}\right],\ \text{with } \omega := \frac{1}{s},\ j = 1, 2.$$

Thus we have

$$1 \Leftrightarrow \frac{1 \Leftrightarrow \beta_j\{s\}}{\beta_j s} \sim (\beta_j s)^{\nu-1} L(1/\beta_j s).$$

In determining $\Delta(\cdot)$, we have taken $L(\cdot) \equiv 1$. Put, cf. (97), for $\rho_2 \in (0, 1 \Leftrightarrow \rho_1)$ (with HT denoting Heavy Traffic),

$$1 \Leftrightarrow W_{HT}(t) := 1 \Leftrightarrow R_{\nu-1}(\Delta(\rho_2)t/\beta_1).$$

As proved in Theorem 1, we have

$$1 \Leftrightarrow W_2(t) \sim \frac{(\nu \Leftrightarrow 1)K}{,\ (1 \Leftrightarrow \nu)(1 \Leftrightarrow \rho)}(t/\beta_1)^{1-\nu} L(t/\beta_1),\ t \to \infty,$$

where K is given in Theorem 3. Define

$$1 \Leftrightarrow W_{RV}(t) := \frac{(\nu \Leftrightarrow 1)K}{,\ (1 \Leftrightarrow \nu)(1 \Leftrightarrow \rho)}(t/\beta_1)^{1-\nu} L(t/\beta_1).$$

From (72) we obtain that $1 \Leftrightarrow W_{HT}(t)$ also exhibits the same asymptotic behavior:

$$1 \Leftrightarrow W_{HT}(t) \sim \frac{(\nu \Leftrightarrow 1)K}{,\ (1 \Leftrightarrow \nu)(1 \Leftrightarrow \rho)}(t/\beta_1)^{1-\nu}L(t/\beta_1).$$

As observed in the previous section, we can also use $1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t)$ to approximate $1 \Leftrightarrow W_2(t)$ where $W(t)$ is the waiting time distribution in the same $M/G/1$ model without priority structure, i.e., the $M/G/1$ queue with FCFS discipline. As remarked there, $1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t)$ has the same tail behavior as $1 \Leftrightarrow W_2(t)$ in the regularly varying case. Therefore, for $0 < \rho < 1$:

$$1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t) \sim 1 \Leftrightarrow W_2(t) \sim 1 \Leftrightarrow W_{HT}(t) \sim 1 \Leftrightarrow W_{RV}(t),\ t \to \infty.$$

We have tested the approximations $1 \Leftrightarrow W_{HT}(t)$, $1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t)$ and $1 \Leftrightarrow W_{RV}(t)$ for three cases: (i) the class-1 service time distribution $B_1(t)$ is specified by (98) and the class-2 service time distribution $B_2(t)$ is exponentially distributed with mean 1; (ii) the class-1 service time distribution $B_1(t)$ is exponentially distributed with mean 1 and the class-2 service time distribution $B_2(t)$ is specified by (98); (iii) both of the class-1 and class-2 service time distributions are specified by (98) with $\nu_1 = \nu_2$.

In view of the very large number of parameter combinations, we have decided to only indicate maximal relative errors over certain t-regions; detailed numerical results can be obtained from the authors. We have distinguished three t-regions: "small" t indicates t-values such that $0.1\rho < W_2(t) \leq 0.5\rho$; "medium" t indicates t-values such that $0.01\rho < W_2(t) \leq 0.1\rho$; "large" t indicates t-values such that $W_2(t) \leq 0.01\rho$. Note that $W_{HT}(0) = 0$ while $W_2(0) = 1 \Leftrightarrow \rho$, so that t-values close to zero always yield large errors. We compare the errors of $1 \Leftrightarrow W_{HT}(t)$, $1 \Leftrightarrow W_{RV}(t)$ and $1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t)$; the latter is referred to as the FCFS case. In the error columns, we consider the absolute value of the largest relative error in a region. Let "⇔⇔⇔" denote that this largest error exceeds 20%; "⇔⇔" that it is between 10% and 20%; "⇔" that it is between 5% and 10%; "+" that it is between 1% and 5%; "++" that it is between 0.1% and 1%; "+ + +" that it is less than 0.1%. Denote by "exp/RV" Case (i), by "RV/exp" Case (ii) and by "RV/RV" Case (iii).

The numerical results are gathered in Table 1-6. Table 1 considers cases with $\rho = 0.9$ and one of the service time distribution being given by (98) and another service time being exponentially distributed with mean 1 or both service time distributions being given by (98) with $\nu = 1.25$. Table 2 does the same except that $\nu = 1.75$. Table 3 considers cases with $\rho = 0.5$ and $\nu = 1.25$; Table 4 does the same except that $\nu = 1.75$; Table 5 consider cases with $\rho = 0.1$ and $\nu = 1.25$; Table 6 consider cases with $\rho = 0.1$ and $\nu = 1.75$.

Main conclusions from the numerical work

1. All the approximations $1 \Leftrightarrow W_{HT}(t)$, $1 \Leftrightarrow W_{RV}(t)$ and $1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t)$ provide extremely accurate approximations for t large.

2. $1 \Leftrightarrow W_{HT}(t)$ performs much better for $\nu = 5/4$ (the case with a heavier tail) than for $\nu = 7/4$.

3. $1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t)$ provides a very good approximation even for small t, better than $1 \Leftrightarrow W_{HT}(t)$ and $1 \Leftrightarrow W_{RV}(t)$; when ρ_1 is small, it performs the best.

4. In heavy traffic (ρ is sufficiently large), $1 \Leftrightarrow W_{HT}(t)$ yields much better results than $1 \Leftrightarrow W_{RV}(t)$ does; $1 \Leftrightarrow W_{RV}(t)$ is almost useless here (it is not a heavy-traffic approximation).

5. In light traffic, $1 \Leftrightarrow W_{HT}(t)$ still provides surprisingly accurate results, when t is not too small and ν is small.

6. In the case of $\nu = 1.75$ and light traffic, the accuracy of $1 \Leftrightarrow W_{HT}(t)$ is almost the same as that of $1 \Leftrightarrow W_{RV}(t)$ or even worse.

Remark 4. If ρ_1 in Case (i) equals ρ_2 in Case (ii) and ρ in both Cases (i) and (ii) are the same, i.e., both cases have the same traffic load for the class with heavy-tailed service time distribution and the same total traffic load, then they have the same contraction coefficients and thus the approximation $1 \Leftrightarrow W_{HT}(t)$ is exactly the same.

Remark 5. $1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t)$ performs particularly well for ρ_1 small, because $\lim_{\rho_1 \to 0}(1 \Leftrightarrow W_2(t))/(1 \Leftrightarrow W((1 \Leftrightarrow \rho_1)t) = 1$. When ρ_1 is small, the busy period of class-1 customers will not have much effect on the class-2 waiting time distribution.

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Table 1: Approximations for class-2 waiting time tails; $\rho = 0.9,\ \nu = 1.25$

	"small" t			"medium" t			"large" t		
	HT	RV	FCFS	HT	RV	FCFS	HT	RV	FCFS
exp/RV:(0.8, 0.1)	$--$	$---$	$+$	$+++$	$--$	$+++$	$+++$	$+$	$+++$
exp/RV:(0.45, 0.45)	$++$	$---$	$+++$	$+++$	$--$	$+++$	$+++$	$+$	$+++$
exp/RV:(0.1, 0.8)	$+++$	$---$	$+++$	$+++$	$--$	$+++$	$+++$	$+$	$+++$
RV/exp:(0.8, 0.1)	$+$	$---$	$+$	$++$	$-$	$++$	$+++$	$+$	$+++$
RV/exp:(0.45, 0.45)	$+$	$---$	$+$	$++$	$-$	$++$	$+++$	$+$	$+++$
RV/exp:(0.1, 0.8)	$--$	$---$	$+$	$++$	$-$	$++$	$+++$	$++$	$+++$
RV/RV:(0.8, 0.1)	$+$	$---$	$+$	$++$	$--$	$++$	$+++$	$+$	$+++$
RV/RV:(0.45, 0.45)	$+$	$---$	$+$	$+$	$-$	$++$	$+++$	$+$	$+++$
RV/RV:(0.1, 0.8)	$++$	$---$	$++$	$+++$	$-$	$+++$	$+++$	$+$	$+++$

Table 2: Approximations for class-2 waiting time tails; $\rho = 0.9,\ \nu = 1.75$

	"small" t			"medium" t			"large" t		
	HT	RV	FCFS	HT	RV	FCFS	HT	RV	FCFS
exp/RV:(0.8, 0.1)	$---$	$---$	$--$	$++$	$-$	$++$	$+++$	$++$	$+++$
exp/RV:(0.45, 0.45)	$---$	$--$	$++$	$--$	$--$	$++$	$++$	$+$	$+++$
exp/RV:(0.1, 0.8)	$---$	$--$	$+++$	$-$	$--$	$+++$	$++$	$+$	$+++$
RV/exp:(0.8, 0.1)	$---$	$--$	$-$	$+$	$--$	$+$	$++$	$+$	$++$
RV/exp:(0.45, 0.45)	$---$	$--$	$+$	$-$	$--$	$++$	$++$	$+$	$++$
RV/exp:(0.1, 0.8)	$-$	$--$	$++$	$++$	$+$	$++$	$+++$	$++$	$+++$
RV/RV:(0.8, 0.1)	$---$	$--$	$-$	$-$	$--$	$+$	$++$	$+$	$++$
RV/RV:(0.45, 0.45)	$---$	$--$	$++$	$-$	$--$	$++$	$++$	$+$	$++$
RV/RV:(0.1, 0.8)	$---$	$--$	$++$	$-$	$--$	$+++$	$++$	$+$	$+++$

Table 3: Approximations for class-2 waiting time tails; $\rho = 0.5,\ \nu = 1.25$

	"small" t			"medium" t			"large" t		
	HT	RV	FCFS	HT	RV	FCFS	HT	RV	FCFS
exp/RV:(0.4, 0.1)	$---$	$---$	$--$	$+++$	$+$	$+++$	$+++$	$++$	$+++$
exp/RV:(0.25, 0.25)	$+$	$---$	$+++$	$+++$	$+$	$+++$	$+++$	$++$	$+++$
exp/RV:(0.1, 0.4)	$++$	$---$	$+++$	$+++$	$+$	$+++$	$+++$	$++$	$+++$
RV/exp:(0.4, 0.1)	$-$	$---$	$-$	$+$	$--$	$-$	$++$	$++$	$++$
RV/exp:(0.25, 0.25)	$---$	$-$	$+$	$--$	$+$	$++$	$++$	$++$	$+++$
RV/exp:(0.1, 0.4)	$---$	$---$	$++$	$-$	$+$	$+++$	$++$	$++$	$+++$
RV/RV:(0.4, 0.1)	$+$	$---$	$+$	$++$	$+$	$++$	$+++$	$++$	$+++$
RV/RV:(0.25, 0.25)	$+$	$---$	$+$	$++$	$+$	$++$	$+++$	$++$	$+++$
RV/RV:(0.1, 0.4)	$+$	$---$	$++$	$++$	$+$	$++$	$+++$	$++$	$+++$

Table 4: Approximations for class-2 waiting time tails; $\rho = 0.5,\ \nu = 1.75$

	"small" t			"medium" t			"large" t		
	HT	RV	FCFS	HT	RV	FCFS	HT	RV	FCFS
exp/RV:(0.4, 0.1)	$-$	$---$	$-$	$+$	$--$	$-$	$++$	$++$	$++$
exp/RV:(0.25, 0.25)	$---$	$---$	$+$	$+$	$+$	$+++$	$++$	$++$	$+++$
exp/RV:(0.1, 0.4)	$---$	$---$	$++$	$-$	$+$	$++$	$++$	$+$	$++$
RV/exp:(0.4, 0.1)	$---$	$---$	$-$	$--$	$+$	$++$	$++$	$+$	$++$
RV/exp:(0.25, 0.25)	$---$	$---$	$-$	$--$	$+$	$+$	$++$	$+$	$++$
RV/exp:(0.1, 0.4)	$-$	$---$	$+$	$-$	$-$	$+$	$++$	$+$	$++$
RV/RV:(0.4, 0.1)	$---$	$---$	$-$	$-$	$+$	$+$	$+++$	$+$	$++$
RV/RV:(0.25, 0.25)	$---$	$--$	$+$	$--$	$+$	$++$	$+++$	$+$	$++$
RV/RV:(0.1, 0.4)	$---$	$---$	$+$	$--$	$+$	$++$	$++$	$+$	$++$

Table 5: Approximations for class-2 waiting time tails; $\rho = 0.1,\ \nu = 1.25$

	"small" t			"medium" t			"large" t		
	HT	RV	FCFS	HT	RV	FCFS	HT	RV	FCFS
exp/RV:(0.09, 0.01)	$---$	$---$	$+$	$--$	$--$	$-$	$+++$	$+++$	$+++$
exp/RV:(0.05, 0.05)	$-$	$-$	$++$	$++$	$++$	$++$	$+++$	$+++$	$+++$
exp/RV:(0.01, 0.09)	$+$	$-$	$+++$	$++$	$++$	$+++$	$+++$	$++$	$+++$
RV/exp:(0.09, 0.01)	$+$	$-$	$+$	$++$	$++$	$++$	$+++$	$+++$	$+++$
RV/exp:(0.05, 0.05)	$---$	$---$	$++$	$++$	$++$	$++$	$+++$	$+++$	$+++$
RV/exp:(0.01, 0.09)	$---$	$---$	$++$	$++$	$+$	$+++$	$+++$	$+++$	$+++$
RV/RV:(0.09, 0.01)	$+$	$+$	$++$	$++$	$++$	$++$	$+++$	$+++$	$+++$
RV/RV:(0.05, 0.05)	$-$	$-$	$++$	$+++$	$++$	$+++$	$+++$	$++$	$+++$
RV/RV:(0.01, 0.09)	$+$	$-$	$+++$	$+++$	$++$	$+++$	$+++$	$++$	$+++$

Table 6: Approximations for class-2 waiting time tails; $\rho = 0.1,\ \nu = 1.75$

	"small" t			"medium" t			"large" t		
	HT	RV	FCFS	HT	RV	FCFS	HT	RV	FCFS
exp/RV:(0.09, 0.01)	$---$	$---$	$--$	$---$	$---$	$--$	$+$	$+$	$++$
exp/RV:(0.05, 0.05)	$---$	$---$	$+$	$-$	$-$	$++$	$++$	$++$	$+++$
exp/RV:(0.01, 0.09)	$---$	$---$	$+++$	$+$	$+$	$+++$	$++$	$+++$	$+++$
RV/exp:(0.09, 0.01)	$---$	$---$	$+$	$-$	$-$	$++$	$++$	$++$	$++$
RV/exp:(0.05, 0.05)	$---$	$---$	$+$	$-$	$-$	$++$	$++$	$+++$	$+++$
RV/exp:(0.01, 0.09)	$---$	$---$	$++$	$+$	$+$	$++$	$++$	$++$	$+++$
RV/RV:(0.09, 0.01)	$---$	$---$	$+$	$+$	$+$	$++$	$+++$	$+++$	$+++$
RV/RV:(0.05, 0.05)	$---$	$---$	$++$	$+$	$+$	$++$	$++$	$+++$	$+++$
RV/RV:(0.01, 0.09)	$---$	$---$	$++$	$+$	$+$	$++$	$+++$	$+++$	$+++$